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On Abstract Conditional Expectations

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1. INTRODUCTION

Let $(\Omega, \hat{\Sigma}, \mu)$ be a probability space and X a Banach space. The following results are well known for abstract conditional expectations

(i) If Σ is a sub-σ-field of $\hat{\Sigma}$, $f_n(\omega) \to^s f(\omega)$ μ-a.e. and there is a non-negative real valued function $\varphi \in L^1(\Omega)$ s.t. $\|f_n(\omega)\| \leqslant \varphi(\omega)$ μ-a.e. for all $n \geqslant 1$ then $\lim_{n\to\infty} E^{\Sigma} f_n(\omega) = E^{\Sigma} f(\omega)$. This is just the Lebesgue dominated convergence theorem for abstract conditional expectations (see [5]).

(ii) If $\{\Sigma_n, \Sigma\}_{n \geqslant 1}$ are sub-σ-fields of $\hat{\Sigma}$, $\bigvee_{n=1}^{\infty} \Sigma_n = \Sigma$ or $\bigcap_{n=1}^{\infty} \Sigma_n = \Sigma$ and $f \in L^p_X(\Omega)$ $(1 \leqslant p < \infty)$ then $E^{\Sigma_n} \to^{L^p_X(\Omega)} E^{\Sigma} f$. Furthermore if $p = 1$ then $E^{\Sigma_n} f(\omega) \to^s E^{\Sigma} f(\omega)$ μ-a.e. These are known as martingale convergence theorems (see [5] and [14, Theorem 5]). Together with the general theory of martingales have an increasingly important impact in Banach space theory (see [5, Chap. V]).

In this note we investigate what happens when $\{\Sigma_n\}_{n \geqslant 1}$ is not a monotone sequence of sub-σ-fields, but rather it converges to Σ in a more complicated way. Namely, $\Sigma = \bigvee_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \Sigma_m = \bigcap_{n=1}^{\infty} \bigvee_{m=n}^{\infty} \Sigma_m$ or using the Kuratowski terminology $\Sigma = \liminf_{n\to\infty} \Sigma_n = \limsup_{n\to\infty} \Sigma_n$. Previous work on this issue was done for $\mathbb{R}$-valued functions by Fetter [6]. Using completely different techniques, that depended on the fact that $X = \mathbb{R}$, she managed to show that $\{E^{\Sigma_n} f\}_{n \geqslant 1}$ converges to $E^{\Sigma} f$ in μ-measure and using that she finally proved that we also have L^p-convergence. Here we first prove that $\{E^{\Sigma_n} f\}_{n \geqslant 1}$ converges to $E^{\Sigma} f$ in the $L^p_X(\Omega)$ $(1 \leqslant p < \infty)$-norm and so we get as a corollary of that the convergence in measure result. Then we consider the case where $f(\cdot)$ is substituted by a sequence $\{f_n(\cdot)\}_{n \geqslant 1} \subseteq L^p_X(\Omega)$ $(1 \leqslant p < \infty)$. We prove two convergence theorems, one for the case where $f_n \to f$ weakly in $L^p_X(\Omega)$ and another for the case where $f_n \to f$ strongly in $L^p_X(\Omega)$. Finally in the last section the above results are extended to compact convex valued measurable multifunctions. We prove

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convergence results for both the Hausdorff metric and the weaker notion of Kuratowski–Mosco convergence.

For the necessary background on Bochner integration and the vector valued conditional expectation we refer to [5]. In the next section we will recall a few facts about measurable multifunctions, their conditional expectation and their different modes of convergence. Details can be found in Castaing and Valadier [2], Himmelberg [8], Rockafellar [10, 11] and Salinetti and Wets [12, 13]. This material will be needed in Section 4.

2. Measurable Multifunctions

Let $F: \Omega \to 2^X\{\varnothing\}$ be a multivalued (set valued) function from the space Ω to the family of subsets of a space X. The set

$$\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X: x \in F(\omega)\}$$

is called the graph of $F(\cdot)$. Also if $V \subseteq X$ we define the following set

$$F^-(V) = \{\omega \in \Omega: F(\omega) \cap V \neq \varnothing\}.$$

This is known as the weak inverse image of V by $F(\cdot)$.

When X is a topological vector space, by $P_f(X)$ (resp. $P_k(X)$) we will denote the nonempty, closed (resp. compact) subsets of X. If X is a locally convex space, then a w in front of f (resp. k) will mean that the closedness (resp. compactness) is with respect to the weak topology on X. Finally a c after f or k will mean that the set is in addition convex.

In the next theorem we summarize the major results concerning the measurability of a multifunction.

Theorem 2.1. *Let (Ω, Σ) be a measurable space and X a separable metric space. Let $F: \Omega \to P_f(X)$ be a multifunction. Consider the following statements*

(1) *$F^-(B) \in \Sigma$ for every $B \in B(X)$ the Borel σ-field of X.*

(2) *$F^-(C) \in \Sigma$ for every C a closed subset of X.*

(3) *$F^-(U) \in \Sigma$ for every U an open subset of X.*

(4) *$\omega \to d(x, F(\omega))$ is a measurable function for every $x \in X$.*

(5) *There exists a sequence of measurable selectors $f_n(\cdot)$ of $F(\cdot)$ s.t. $\operatorname{cl}\{f_n(\omega)\}_{n\geq 1} = F(\omega)$ (Castaing representation).*

(6) $\operatorname{Gr} F \in \Sigma \times B(X)$.

Then we have the following results.

(i) $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (6)$.

(ii) *If X is Polish (i.e., is in addition complete) then* $(3) \Leftrightarrow (5)$.

(iii) *If X is Polish and there is a complete σ-finite measure on Σ then* (1) *to* (6) *are all equivalent.*

Following Himmelberg [8] we say that a multifunction satisfying (2) (resp. (3)) is measurable (resp. weakly measurable).

Given any multifunction $F: \Omega \to P_f(X)$ (X a separable Banach space) we can define the following set $S_F^1 = \{f \in L_X^1(\Omega): f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. This can be empty. If it is nonempty then it is easy to check that it is a closed subset of $L_X^1(\Omega)$.

Using this set we can define an integral of $F(\cdot)$

$$\int_\Omega F(\omega)\, d\mu(\omega) = \left\{\int_\Omega f(\omega)\, d\mu(\omega): f \in S_F^1\right\}.$$

This definition was first introduced by Aumann [1] for $X = \mathbb{R}^n$. It is a generalization of both the integral of single valued functions and of the Minkowski summation of sets. It turned out to be an extremely powerful tool in several areas of applied mathematics, especially in optimal control theory and mathematical economics.

Clearly if $S_F^1 = \varnothing$ then $\int_\Omega F(\omega)\, d\mu(\omega) = \varnothing$. We will say that $F(\cdot)$ is integrably bounded if there is a $\varphi(\cdot) \in L^1(\Omega)$ s.t.

$$|F(\omega)| = \sup_{x \in F(\omega)} \|x\| \leqslant \varphi(\omega) \qquad \mu\text{-a.e.}$$

Using the Kuratowski–Ryll Nardzewski selection theorem (see [10]) we have that in this case $S_F^1 \neq \varnothing$ and so $\int_\Omega F(\omega)\, d\mu(\omega) \neq \varnothing$.

From Hiai and Umegaki [7] we know that for Σ_0 a sub-σ-field of Σ and $F: \Omega \to P_f(X)$ an integrably bounded multifunction, there exists a unique multifunction $E^{\Sigma_0}F: \Omega \to P_f(X)$ which is Σ_0-measurable integrably bounded and we have that

$$S^1_{E^{\Sigma_0}F}(\Sigma_0) = \mathrm{cl}\{E^{\Sigma_0}f: f \in S_F^1\}$$

where the closure is taken in the $L_X^1(\Omega)$-norm. We call $E^{\Sigma_0}F(\cdot)$ the set valued conditional expectation of $F(\cdot)$ with respect to Σ_0.

Recall that for any two sets A, B in X we can define their Hausdorff distance by

$$h(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}.$$

We know that $P_f(X)$ endowed with the Hausdorff distance $h(\cdot, \cdot)$ is a complete metric space (see [2]). When $F_1(\cdot)$ and $F_2(\cdot)$ are two integrably bounded multifunctions from Ω into $P_f(X)$, we set $\Delta(F_1, F_2) = \int_\Omega h(F_1(\omega), F_2(\omega))\, d\mu(\omega)$. This is a metric on all integrably bounded $F\colon \Omega \to P_f(X)$. In fact the metric space is complete.

Another notion of convergence of sets, more appropriate in the study of the stability of optimization and variational problems, is the one introduced by Mosco [9] and extensively studied by Salinetti and Wets [12, 13]. Let X be a Banach space and let $\{A_n\}_{n \geqslant 1}$ be a sequence of non-empty, closed subsets of X. Let t be the topology on X. We say that A_n t-converges in the Kuratowski–Mosco sense to A if

$$t - \overline{\lim_{n \to \infty}}\, A_n \subseteq A \subseteq t - \underline{\lim_{n \to \infty}}\, A_n$$

where

$$t - \overline{\lim_{n \to \infty}}\, A_n = \{x = t - \lim_{m \to \infty} x_m \colon x_m \in A_m \;\; m \in M \subseteq N\}$$

and

$$t - \underline{\lim_{n \to \infty}}\, A_n = \{x = t - \lim_{n \to \infty} x_n \colon x_n \in A_n \;\; n \in N\}.$$

Since we always have that

$$t - \underline{\lim_{n \to \infty}}\, A_n \subseteq t - \overline{\lim_{n \to \infty}}\, A_n$$

we deduce that $\{A_n\}_{n \geqslant 1}$ t-converges to A in the Kuratowski–Mosco sense if and only if $t - \overline{\lim}_{n \to \infty} A_n = A = t - \underline{\lim}_{n \to \infty} A_n$ and in that case we write that $A_n \to^{t-K-M} A$ as $n \to \infty$.

When $w - \overline{\lim}_{n \to \infty} A_n = A = s - \underline{\lim}_{n \to \infty} A_n$ (where w denotes the weak topology and s the strong topology), we will say that A_n converges to A in the Kuratowski–Mosco sense and write $A_n \to^{K-M} A$ as $n \to \infty$.

Finally if $\{f_n\}_{n \geqslant 1}$ is a sequence of closed, convex functions we will say that $f_n \to^{\tau} f$ as $n \to \infty$ if and only if $\mathit{epi} f_n \to^{K-M} \mathit{epi} f$ as $n \to \infty$.

3. Convergence Results for Single Valued Functions

In the next result let X be a Banach space having the Radon–Nikodym property (see [5]). Let $(\Omega, \hat{\Sigma}, \mu)$ be a complete probability space and let $\{\Sigma_n\}_{n \geqslant 1}$ be sub-σ-fields of $\hat{\Sigma}$.

As already mentioned in Section 1, $\liminf_{n \to \infty} \Sigma_n = \bigvee_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \Sigma_m$ and $\limsup_{n \to \infty} \Sigma_n = \bigcap_{n=1}^{\infty} \bigvee_{m=n}^{\infty} \Sigma_m$.

Our first convergence result is the following.

THEOREM 3.1. *If* $\limsup_{n\to\infty} \Sigma_n = \liminf_{n\to\infty} \Sigma_n = \Sigma$ *and* $f \in L^p_X(\Omega)$ $(1 \leqslant p < \infty)$ *then*

$$E^{\Sigma_n} f \xrightarrow{L^p_X(\Omega)} E^{\Sigma} f.$$

Proof. We know (see [5, p. 123]) that for all $n \geqslant 1$

$$\|E^{\Sigma_n} f\|_p \leqslant \|f\|_p \quad \text{and} \quad \|E^{\Sigma} f\|_p \leqslant \|f\|_p.$$

Let $\Lambda_n = \bigcap_{m\geqslant n} \Sigma_m$ and $K_n = \bigvee_{m\geqslant n} K_m$. Clearly for all $n \geqslant 1$, $\Lambda_n \subseteq \Sigma_n \subseteq K_n$ and furthermore $\Lambda_n \uparrow \Sigma$, $K_n \downarrow \Sigma$ as $n \to \infty$. From Minkowski's inequality we have that

$$\|E^{\Sigma_n} f - E^{\Sigma} f\|_p \leqslant \|E^{\Sigma_n} f - E^{\Lambda_n} f\|_p + \|E^{\Lambda_n} f - E^{\Sigma} f\|_p. \tag{$*$}$$

Since $\Lambda_n \uparrow \Sigma$ as $n \to \infty$ we know from Chatterji's martingale convergence theorem (see [3, 5, p. 126 Corollary 4]) that $E^{\Lambda_n} f \to^{L^p_X(\Omega)} E^{\Sigma} f$. So we have that $\lim_{n\to\infty} \|E^{\Lambda_n} f - E^{\Sigma} f\|_p = 0$. Hence we need to estimate $\|E^{\Sigma_n} f - E^{\Lambda_n} f\|_p$. Using the fact that for all $n \geqslant 1$, $\Lambda_n \subseteq \Sigma_n \subseteq K_n$ we have that

$$\begin{aligned}\|E^{\Sigma_n} f - E^{\Lambda_n} f\|_p &= \|E^{\Sigma_n} f - E^{\Sigma_n} E^{\Lambda_n} f\|_p = \|E^{\Sigma_n}(f - E^{\Lambda_n} f)\|_p \\ &\leqslant \|E^{K_n}(f - E^{\Lambda_n} f)\|_p = \|E^{K_n} f - E^{\Lambda_n} f\|_p.\end{aligned}$$

Note that

$$\|E^{K_n} f - E^{\Lambda_n} f\|_p \leqslant \|E^{K_n} f - E^{\Sigma} f\|_p + \|E^{\Sigma} f - E^{\Lambda_n} f\|_p$$

and once again from Chatterji's martingale convergence theorem we have that $\|E^{K_n} f - E^{\Sigma} f\|_p \to 0$ and $\|E^{\Sigma} f - E^{\Lambda_n} f\|_p \to 0$ as $n \to \infty$.

Hence $\|E^{K_n} f - E^{\Lambda_n} f\|_p \to 0$ as $n \to \infty$ which means that $\|E^{\Sigma_n} f - E^{\Lambda_n} f\|_p \to 0$ as $n \to \infty$. Going back to $(*)$ we finally have that

$$\|E^{\Sigma_n} f - E^{\Sigma} f\|_p \to 0$$

as $n \to \infty$. Q.E.D.

Recall that because of Chebyshev's inequality $L^p_X(\Omega)$-convergence implies convergence in measure. So we have the following corollary of Theorem 3.1. Assume that the hypotheses of the theorem hold.

COROLLARY. $E^{\Sigma_n} f(\cdot) \to^{\mu} E^{\Sigma} f(\cdot)$ *as* $n \to \infty$.

Theorem 3.1 can be generalized. Instead of $f \in L^p_X(\Omega)$ consider a sequence $\{f_n, f\}_{n\geqslant 1} \subseteq L^p_X(\Omega)$. We will first consider the case where $f_n \to^{w-L^p_X(\Omega)} f$. For that purpose we need the following lemma, which is interesting for its own sake. Assume that X is a Banach space and that X^* has the Radon–Nikodym (R–N) property. Also Σ is a sub-σ-field of $\hat{\Sigma}$.

LEMMA 3.1. *If $f \in L^p_X(\Omega)$ and $g \in [L^p_X(\Omega, \Sigma)]^*$ $(1 \leqslant p < \infty)$ then*

$$\int_\Omega (f(\omega), g(\omega))\, d\mu(\omega) = \int_\Omega (E^\Sigma f(\omega), g(\omega))\, d\mu(\omega).$$

Proof. First note that because X^* has the R–N property $[L^p_X(\Omega, \Sigma)]^* = L^q_{X^*}(\Omega, \Sigma)$ where $1/p + 1/q = 1$ (see [5, p. 98, Theorem 1]). Next let $g = x^* \chi_A$, where $A \in \Sigma$. Then we have that

$$\int_\Omega (f(\omega), g(\omega))\, d\mu(\omega) = \int_A (f(\omega), x^*)\, d\mu(\omega) = \left(x^*, \int_A f(\omega)\, d\mu(\omega) \right)$$
$$= \left(x^*, \int_A E^\Sigma f(\omega)\, d\mu(\omega) \right)$$
$$= \int_\Omega (E^\Sigma f(\omega), g(\omega))\, d\mu(\omega).$$

Next let $g(\omega) = \sum_{n=1}^N x_n^* \chi_{A_n}(\omega)$, $A_n \in \Sigma$, $n \geqslant 1$. Then

$$\int_\Omega (f(\omega), g(\omega)\, d\mu(\omega) = \sum_{n=1}^N \int_\Omega (f(\omega), x_n^* \chi_{A_n}(\omega))\, d\mu(\omega)$$
$$= \sum_{n=1}^N \int_{A_n} (f(\omega), x_n^*)\, d\mu(\omega)$$
$$= \sum_{n=1}^N \int_\Omega (E^\Sigma f(\omega), x_n^* \chi_{A_n}(\omega)\, d\mu(\omega)$$
$$= \int_\Omega (E^\Sigma f(\omega), g(\omega))\, d\mu(\omega).$$

For arbitrary $g(\cdot) \in L^q_{X^*}(\Omega, \Sigma)$ we can find simple functions $\{s_n\}_{n \geqslant 1}$ s.t. $s_n(\omega) \to^s g(\omega)$ μ-a.e. and $\|s_n(\omega)\| \leqslant \varphi(\omega)$ μ-a.e. where $\varphi \in L^1_+(\Omega, \Sigma)$. Then we have

$$\int_\Omega (f(\omega), s_n(\omega)\, d\mu(\omega) = \int_\Omega (E^\Sigma f(\omega), s_n(\omega))\, d\mu(\omega).$$

Applying Lebesgue's dominated convergence theorem we finally get that

$$\int_\Omega (f(\omega), g(\omega))\, d\mu(\omega) = \int_\Omega (E^\Sigma f(\omega), g(\omega))\, d\mu(\omega)$$

as claimed. Q.E.D.

We can use that to obtain the following convergence result. The hypotheses on X used in Lemma 3.1 remain in effect. Furthermore by $\langle \cdot, \cdot \rangle$ we will denote the duality brackets between $L^p_X(\Omega)$ and $L^q_{X^*}(\Omega)$. Also assume that X has the R–N property too. Finally $\Lambda_n = \bigcap_{m=n}^{\infty} \Sigma_m$.

THEOREM 3.2. *If* $\limsup_{n\to\infty} \Sigma_n = \liminf_{n\to\infty} \Sigma_n = \Sigma$ and $f_n \xrightarrow{w-L^p_X} f$ $(1 \leqslant p < \infty)$ *then for all* $g \in L^q_{X^*}(\Omega, \Lambda_1)$, $\langle g, E^{\Sigma_n} f_n \rangle \to \langle g, E^{\Sigma} f \rangle$ *as* $n \to \infty$.

Proof. For all $n \geqslant 1$ we have that

$$\begin{aligned}\langle g, E^{\Sigma_n} f_n - E^{\Sigma} f \rangle &= \langle g, E^{\Sigma_n} f_n - E^{\Sigma_n} f + E^{\Sigma_n} f - E^{\Sigma} f \rangle \\ &= \langle g, E^{\Sigma_n}(f_n - f) \rangle + \langle g, (E^{\Sigma_n} - E^{\Sigma}) f \rangle.\end{aligned}$$

From Theorem 3.1 we know that $E^{\Sigma_n} f \xrightarrow{L^p_X(\Omega)} E^{\Sigma} f$ as $n \to \infty$. Hence $\langle g, (E^{\Sigma_n} - E^{\Sigma}) f \rangle \to 0$ as $n \to \infty$. Also since by hypothesis $f_n \xrightarrow{w-L^p_X(\Omega)} f$ we have that $\langle g, f_n - f \rangle \to 0$ as $n \to \infty$. Using Lemma 3.1 we get that

$$\langle g, E^{\Sigma_n} f_n - E^{\Sigma} f \rangle \to 0 \qquad \text{as} \quad n \to \infty. \qquad \text{Q.E.D.}$$

Finally we will consider the case where the f_n's converge strongly in $L^p_X(\Omega)$ $(1 \leqslant p < \infty)$ to f. Here we assume that X is a Banach space having the Radon–Nikodym property.

THEOREM 3.3. *If* $\limsup_{n\to\infty} \Sigma_n = \liminf_{n\to\infty} \Sigma_n = \Sigma$ *and* $f_n \xrightarrow{L^p_X(\Omega)} f$ *as* $n \to \infty$. *Then* $E^{\Sigma_n} f_n \xrightarrow{L^p_X(\Omega)} E^{\Sigma} f$ *as* $n \to \infty$.

Proof. The Minkowski inequality tells us that

$$\begin{aligned}\|E^{\Sigma_n} f_n - E^{\Sigma} f\|_p &= \|E^{\Sigma_n} f_n - E^{\Sigma_n} f + E^{\Sigma_n} f - E^{\Sigma} f\|_p \\ &\leqslant \|E^{\Sigma_n} f_n - E^{\Sigma_n} f\|_p + \|E^{\Sigma_n} f - E^{\Sigma} f\|_p.\end{aligned}$$

From Theorem 3.1 we know that $\|E^{\Sigma_n} f - E^{\Sigma} f\|_p \to 0$ as $n \to \infty$. Also recall that the abstract conditional expectation is a contraction. So we get that

$$\|E^{\Sigma_n} f_n - E^{\Sigma_n} f\|_p \leqslant \|f_n - f\|_p \to 0 \qquad \text{as} \quad n \to \infty.$$

Therefore we conclude that $\|E^n f_n - Ef\|_p \to 0$ as $n \to \infty$. Q.E.D.

4. CONVERGENCE THEOREMS FOR MULTIFUNCTIONS

Throughout this section all σ-fields will be complete with respect to the probability measure μ. So let $(\Omega, \hat{\Sigma}, \mu)$ be a complete probability space, $\{\Sigma_n\}_{n \geqslant 1}$ a sequence of sub-σ-fields of $\hat{\Sigma}$ and X a separable Banach space, having the R–N property.

We start with a convergence result that involves the Hausdorff metric.

THEOREM 4.1. *If $F: \Omega \to P_{kc}(X)$ is integrably bounded and*

$$\limsup_{n \to \infty} \Sigma_n = \liminf_{n \to \infty} \Sigma_n = \Sigma$$

then $\Delta(E^{\Sigma_n}F, E^{\Sigma}F) \to 0$ as $n \to \infty$ and if $|E^{\Sigma_n}F(\omega)| \leqslant \psi(\omega)$, where $\psi(\cdot) \in L^1_+(\Omega)$, $n \geqslant 1$ then $E^{\Sigma_n}F(\omega) \to^h E^{\Sigma}F(\omega)$ μ-a.e.

Proof. Using the triangle inequality for the Hausdorff metric we have for any $\omega \in \Omega$ and all $n \geqslant 1$

$$h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega)) \leqslant h(E^{\Sigma_n}F(\omega), E^{\Lambda_n}F(\omega)) + h(E^{\Lambda_n}F(\omega), E^{\Sigma}F(\omega)).$$

Integrating both sides we get that

$$\int_\Omega h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega))\, d\mu(\omega) \leqslant \int_\Omega h(E^{\Sigma_n}F(\omega), E^{\Lambda_n}F(\omega))\, d\mu(\omega)$$

$$+ \int_\Omega h(E^{\Lambda_n}F(\omega), E^{\Sigma}f(\omega))\, d\mu(\omega). \quad (*)$$

From Theorem 6.1 of [7] we know that $\int_\Omega h(E^{\Lambda_n}F(\omega), E^{\Sigma}F(\omega))\, d\mu(\omega) \to 0$ as $n \to \infty$. Also since for $n \geqslant 1$, $\Lambda_n \subseteq \Sigma_n \subseteq K_n$ we have that

$$h(E^{\Sigma_n}F(\omega), E^{\Lambda_n}F(\omega)) = h(E^{\Sigma_n}E^{K_n}F(\omega), E^{\Sigma_n}E^{\Lambda_n}F(\omega))$$

and using Theorem 5.2(1°) of [7] we get that

$$\int_\Omega h(E^{\Sigma_n}E^{K_n}F(\omega), E^{\Sigma_n}E^{\Lambda_n}F(\omega))\, d\mu(\omega)$$

$$\leqslant \int_\Omega h(E^{K_n}F(\omega), E^{\Lambda_n}F(\omega))\, d\mu(\omega).$$

But note that

$$\int_\Omega h(E^{K_n}F(\omega), E^{\Lambda_n}F(\omega))\, d\mu(\omega)$$

$$= \Delta(E^{K_n}F, E^{\Lambda_n}F) \leqslant \Delta(E^{K_n}F, E^{\Sigma}F) + \Delta(E^{\Sigma}F, E^{\Lambda_n}F)$$

and again by Theorem 6.1 of [7] we know that $\Delta(E^{K_n}F, E^{\Sigma}F) \to 0$ and $\Delta(E^{\Sigma}F, E^{\Lambda_n}F) \to 0$ as $n \to \infty$. So we have that $\int_\Omega h(E^{\Sigma_n}F(\omega), E^{\Lambda_n}F(\omega))\, d\mu(\omega) \to 0$ as $n \to \infty$ and going back to $(*)$ we get that

$$\Delta(E^{\Sigma_n}F, E^{\Sigma}F) = \int_\Omega h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega))\, d\mu(\omega) \to 0 \qquad \text{as} \quad n \to \infty.$$

So $h(E^{\Sigma_n}F(\cdot), E^{\Sigma}F(\cdot)) \to 0$ in $L^1(\Omega)$. Hence $h(E^{\Sigma_n}F(\cdot), E^{\Sigma}F(\cdot)) \to^{\mu} 0$ as $n \to \infty$.

Next note that for all $A \in \hat{\Sigma}$

$$\int_A h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega))\, d\mu(\omega) \leqslant \int_{\Omega} h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega))\, d\mu(\omega).$$

Also for all $n \geqslant 1$

$$\begin{aligned} h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega)) &\leqslant h(E^{\Sigma_n}F(\omega) - E^{\Sigma}F(\omega), 0) \\ &= |E^{\Sigma_n}F(\omega) - E^{\Sigma}F(\Omega)| \\ &\leqslant \psi(\omega) + E^{\Sigma}\varphi(\omega) \qquad \mu\text{-a.e.} \end{aligned}$$

where $\varphi(\cdot)$ is the $L^1(\Omega)$-bound of $F(\cdot)$. So applying the Lebesgue dominated convergence theorem we get that

$$\int_A \lim_{n \to \infty} h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega))\, d\mu(\omega) = 0$$

for all $A \in \hat{\Sigma}$. Therefore $\lim_{n \to \infty} h(E^{\Sigma_n}F(\omega), E^{\Sigma}F(\omega)) = 0$ μ-a.e. Q.E.D.

In the above theorem our assumption on $F(\cdot)$ was very strong. We can relax it and obtain another pointwise convergence theorem for the strong Kuratowski–Mosco convergence. So assume that X^* has the Radon–Nikodym property too.

THEOREM 4.2. *If $F: \Omega \to P_{wkc}(X)$ is integrably bounded,* $\limsup_{n \to \infty} \Sigma_n = \liminf_{n \to \infty} \Sigma_n = \Sigma$ *and for all $n \geqslant 1$, $|E^{\Sigma_n}F(\omega)| \leqslant \psi(\omega)$ μ-a.e. where $\psi(\cdot) \in L^1(\Omega)$ then $E^{\Sigma_n}F(\omega) \to^{s-K-M} E^{\Sigma}F(\omega)$ μ-a.e.*

Remark. In this theorem $F(\cdot)$ takes w-compact, convex values which is a significant generalization over the s-compact, convex values that it had in Theorem 4.1. For example in a reflexive Banach space the unit ball and more generally any w-closed, bounded set is w-compact (Alaoglu's theorem), but it is not s-compact. In fact s-compact sets in an infinite dimensional Banach space have an empty interior.

Proof. Let $g(\cdot) \in S^1_{E^{\Sigma}F}$. We know from [7, Theorem 5.1] that

$$S^1_{E^{\Sigma}F} = \text{cl}\, E^{\Sigma} S^1_F$$

where the closure is taken with respect to the $L^1_X(\Omega)$-norm. We claim that S^1_F is w-compact in $L^1_X(\Omega)$. A straightforward way to see that is the following. Let $u(\cdot)$ $L^{\infty}_{X^*}(\Omega) = [L^1_X(\Omega)]^*$

$$\sup_{f \in S^1_F} \langle u, f \rangle = \sup_{f \in S^1_F} \int_{\Omega} (u(\omega), f(\omega))\, d\mu(\omega).$$

From Theorem 2.1 of [7] we have that

$$\sup_{f \in S_F^1} \int_\Omega (u(\omega), f(\omega))\, d\mu(\omega) = \int_\Omega \sup_{x \in F(\omega)} (u(\omega), x)\, d\mu(\omega).$$

Because of the w-compactness of $F(\ )$ and thanks to the Kuratowski–Ryll Nardzewski selection theorem we can find $\hat{x}\colon \Omega \to X$ measurable s.t. $\hat{x}(\omega) \in F(\omega)$ for all $\omega \in \Omega$. So $\hat{x}(\cdot) \in S_F^1$. Hence we have that

$$\sup_{f \in S_F^1} \langle u, f \rangle = \langle u, x \rangle$$

and since $u \in L_{X^*}^\infty(\Omega)$ was arbitrary we conclude by James' theorem that S_F^1 is w-compact in $L_X^1(\Omega)$. (See [4, p. 13, Theorem 4]). Since $E^\Sigma(\cdot)$ is a continuous, linear operator we know that it is also w-continuous and so we deduce that $E^\Sigma S_F^1$ is w-compact in $L_X^1(\Omega, \Sigma)$. Also it is convex since S_F^1 is. Recalling that for convex sets weak and strong closure coincide, we finally have that

$$S_{E^\Sigma F}^1 = E^\Sigma S_F^1.$$

So $g = E^\Sigma f$ for some $f \in S_F^1$. Applying Theorem 3.1 we have that for all $A \in \Sigma$ $\lim_{n \to \infty} \int_A E^{\Sigma_n} f(\omega) = \int_A E^\Sigma f(\omega)$. Since by the corollary to Theorem 3.1 $E^{\Sigma_n} f(\ ) \to^\mu E^\Sigma f(\cdot)$ as $n \to \infty$, we can apply Lebesgue's dominated convergence theorem (see [5, p. 45]) to get that

$$\int_A \lim_{n \to \infty} E^{\Sigma_n} f(\omega) = \int_A E^\Sigma f(\omega) \qquad \text{for all } A \in \hat{\Sigma}.$$

So $\lim_{n \to \infty} E^n f(\omega) = E^\Sigma f(\omega)$ μ-a.e. Hence we get that $g(\omega) = E^\Sigma f(\omega) \in s-\underline{\lim}_{n \to \infty} E^{\Sigma_n} F(\omega)$ μ-a.e. Since $g(\cdot)$ was arbitrary we have that this is true for all elements of $S_{E^\Sigma F}^1$. From Castaing's representation result we know that there exists $\{g_n(\cdot)\}_{n \geqslant 1} \subseteq S_{E^\Sigma F}^1$ s.t. $\operatorname{cl}\{g_n(\omega)\}_{n \geqslant 1} = E^\Sigma F(\omega)$. Hence we conclude that

$$E^\Sigma F(\omega) \subseteq s\text{-}\liminf_{n \to \infty} E^{\Sigma_n} F(\omega) \qquad \mu\text{-a.e.} \tag{1}$$

Next consider $\omega \to s\text{-}\limsup_{n \to \infty} E^{\Sigma_n} F(\omega)$. This is a measurable and integrably bounded multifunction. Let $v(\cdot) \in S^1_{s\text{-}\limsup_{n \to \infty} E^{\Sigma_n} F}$. Then

$$\liminf_{n \to \infty} \|v(\omega) - E^{\Sigma_n} F(\omega)\| = 0 \qquad \mu\text{-a.e.}$$

Note that $E^{\Sigma_n} F(\omega) \in P_{wkc}(X)$ for all $\omega \in \Omega$. Hence for all $n \geqslant 1$

$$M_n(\omega) = \{x \in E^{\Sigma_n} F(\omega) \colon \|v(\omega) - x\| = \|v(\omega) - E^{\Sigma_n} F(\omega)\|\}$$

is nonempty and closed for all $\omega \in \Omega$. Apply the Kuratowski–Ryll Nordzewski selection theorem to find measurable functions $m_n : \Omega \to X$ s.t. $m_n(\omega) \in M_n(\omega)$ for all $\omega \in \Omega$, $n \geqslant 1$. Then $m_n = E^{\Sigma_n} f$, $f \in S_F$, $n \geqslant 1$. As above we can get that $\|E^{\Sigma_n} f(\omega) - E^{\Sigma} f(\omega)\| \to 0$ μ-a.e. as $n \to \infty$. Now we note that

$$\|g(\omega) - E^{\Sigma} f(\omega)\| \leqslant \|g(\omega) - E^{\Sigma_n} f(\omega)\| + \|E^{\Sigma_n} f(\omega) - E^{\Sigma} f(\omega)\|$$

and taking lim inf of both sides we have that

$$\|g(\omega) - E^{\Sigma} f(\omega)\| \leqslant \varliminf_{n \to \infty} \|g(\omega) - E^{\Sigma_n} f(\omega)\| + \lim_{n \to \infty} \|E^{\Sigma_n} f(\omega) - E^{\Sigma} f(\omega)\| = 0 \qquad \mu\text{-a.e.}$$

Hence $g(\omega) = E^{\Sigma} f(\omega)$ μ-a.e. which means that $g(\omega) \in E^{\Sigma} F(\omega)$ μ-a.e. and working once again with the Castaing representation we get that

$$s\text{-}\limsup_{n \to \infty} E^{\Sigma_n} F(\omega) \subseteq E^{\Sigma} F(\omega) \qquad \mu\text{-a.e.} \tag{2}$$

From (1) and (2) above we finally have that

$$E^{\Sigma_n} F(\omega) \xrightarrow{s\text{-K-M}} E^{\Sigma} F(\omega) \qquad \mu\text{-a.e.} \qquad \text{Q.E.D.}$$

Using this theorem we can have the following useful result. Our assumptions on X and X^* are the same with those in Theorem 4.2.

THEOREM 4.3. *If* $\limsup_{n \to \infty} \Sigma_n = \liminf_{n \to \infty} \Sigma_n = \Sigma$ *and* $F : \Omega \to P_{wk}(X)$ *is integrably bounded then*

$$\overline{\text{conv}}\, E^{\Sigma_n} F(\omega) \xrightarrow{s\text{-K-M}} \overline{\text{conv}}\, E^{\Sigma} F(\omega) \qquad \mu\text{-a.e.}$$

Proof. From [7] we know that

$$\overline{\text{conv}}\, E^{\Sigma_n} F(\omega) = E^{\Sigma_n}\, \overline{\text{conv}}\, F(\omega) \qquad \mu\text{-a.e.}$$

and

$$\overline{\text{conv}}\, E^{\Sigma} F(\omega) = E^{\Sigma}\, \overline{\text{conv}}\, F(\omega) \qquad \mu\text{-a.e.}$$

Mazur's theorem tells us that $\overline{\text{conv}}\, F(\omega) \in P_{wkc}(X)$ for all $\omega \in \Omega$. Hence we can apply Theorem 4.2 and get that

$$E^{\Sigma_n}\, \overline{\text{conv}}\, F(\omega) \xrightarrow{s\text{-K-M}} E^{\Sigma}\, \overline{\text{conv}}\, F(\omega) \qquad \mu\text{-a.e.}$$

and so

$$\overline{\text{conv}}\, E^{\Sigma_n}F(\omega) \xrightarrow{s\text{-K-M}} \overline{\text{conv}}\, E^{\Sigma}F(\omega) \qquad \mu\text{-a.e.} \qquad \text{Q.E.D.}$$

We can get the stronger convergence in the Hausdorff metric for a sequence of $\{E^{\Sigma_n}F\}_{n \geq 1}$. Assume that X is finite dimensional.

THEOREM 4.4. *If* $\limsup_{n\to\infty} \Sigma_n = \liminf_{n\to\infty} \Sigma_n = \Sigma$ *and* $F: \Omega \to P_{fc}(X)$ *is integrably bounded then*

$$E^{\Sigma_n}F(\omega) \xrightarrow{h} E^{\Sigma}F(\omega) \qquad \mu\text{-a.e.}$$

Proof. From Theorem 5.5 of [7] we can easily deduce that

$$\sigma_{E^{\Sigma_n}F(\omega)}(\cdot) = E^{\Sigma_n}\sigma_{F(\omega)}(\cdot) \qquad \text{and} \qquad \sigma_{E^{\Sigma}F(\omega)}(\cdot) = E^{\Sigma}\sigma_{F(\omega)}(\cdot) \qquad \mu\text{-a.e.}$$

Also from the corollary to Theorem 3.1 we know that

$$E^{\Sigma_n}\sigma_{F(\cdot)}(\cdot) \xrightarrow{\mu} E^{\Sigma}\sigma_{F(\cdot)}(\cdot) \qquad \text{as} \quad n \to \infty.$$

Hence we can find a subsequence $\{n_k\} \subseteq \{n\}$ s.t.

$$E^{\Sigma_{n_k}}\sigma_{F(\omega)}(\cdot) \to E^{\Sigma}\sigma_{F(\omega)}(\cdot) \qquad \mu\text{-a.e.} \quad \text{as} \quad n_k \to \infty.$$

Observe that $\{E^{\Sigma_n}\sigma_{F(\omega)}(\cdot), E^{\Sigma}\sigma_{F(\omega)}(\cdot)\}_{k \geq 1}$, $\omega \in \Omega$, are closed, convex functions which are finite for all $x^* \in X^*$. Hence Corollary 2E of Salinetti and Wets [12] tells us that $E^{\Sigma_n}\sigma_{F(\omega)}(\cdot) \to^{\tau} E^{\Sigma}\sigma_{F(\omega)}(\cdot)$ μ-a.e. But then invoking Theorem 3.1 of Mosco we conclude that

$$E^{\Sigma_n}F(\omega) \xrightarrow{h} E^{\Sigma}F(\omega) \qquad \mu\text{-a.e.} \qquad \text{Q.E.D.}$$

We will close the paper by generalizing Theorems 4.1 and 4.2 to a sequence $\{F_n\}_{n \geq 1}$ of multifunctions.

For the next result assume that the dual X^* has the Radon–Nikodym property too.

THEOREM 4.5. *If* $\limsup_{n\to\infty} \Sigma_n = \liminf_{n\to\infty} \Sigma_n = \Sigma$ *and* $F_n, F: \Omega \to P_{wkc}(X)$ *are integrably bounded by* $\varphi(\cdot) \in L^1(\Omega)$ *and* $F_n(\omega) \to^{\text{K-M}} F(\omega)$ μ*-a.e. then*

$$E^{\Sigma_n}F_n(\omega) \xrightarrow{s\text{-K-M}} E^{\Sigma}F(\omega) \qquad \mu\text{-a.e.}$$

Proof. The proof follows the general pattern of that of Theorem 4.2 so we will skip the details. So let $g \in S^1_F$. Then $g = E^{\Sigma} f$ with $f \in S^1_F$. Using the fact that $\|f(\omega) - F_n(\omega)\| \to 0$ μ-a.e. as $n \to \infty$ together with assumption that for all $n \geq 1$ and all $\omega \in \Omega$, $F_n(\omega) \in P_{wkc}(X)$ and the Kuratowski–Ryll Nardzewski selection theorem, we can find $f_n \in S^1_{E^{\Sigma_n} F_n}$, $n \geq 1$ s.t. $f_n(\omega) \to^s f(\omega)$ μ-a.e. A straightforward application of the Lebesgue's dominated convergence theorem gives us that $E^{\Sigma_n} f_n(\omega) \to^s E^{\Sigma} f(\omega)$ μ-a.e. Since $E^{\Sigma_n} f_n \in S^1_{E^{\Sigma_n} F_n}$ for all $n \geq 1$, we get that $g(\omega) = E^{\Sigma} f(\omega) \in s\text{-}\liminf_{n \to \infty} E^{\Sigma_n} F_n(\omega)$ μ-a.e.

Next let $v(\cdot) \in S^1_{s\text{-}\limsup E^{\Sigma_n} F_n}$. As before we can find $m_n(\cdot) \in S^1_{E^{\Sigma_n} F_n}$ s.t. $\|v(\omega) - m_n(\omega)\| \to 0$ μ-a.e. as $n \to \infty$. Clearly $m_n = E^{\Sigma_n} f_n$ with $f_n \in S^1_{F_n}$, $n \geq 1$ and $E^{\Sigma_n} f(\cdot) \to^{\mu} E^{\Sigma} f(\cdot)$. So $v(\omega) = E^{\Sigma} f(\omega)$ μ-a.e. which means that $v(\omega) \in E^{\Sigma} F(\omega)$ μ-a.e. Hence $E^{\Sigma_n} F_n(\omega) \to^{s\text{-K-M}} E^{\Sigma} F(\omega)$ μ-a.e. Q.E.D.

The corresponding extension of Theorem 4.1 has as follows. As was the case throughout this section, X is separable and has the R–N property.

THEOREM 4.6. *If* $\limsup_{n \to \infty} \Sigma_n = \liminf_{n \to \infty} \Sigma_n = \Sigma$, $F_n: \Omega \to P_{kc}(X)$ *are integrably bounded by* $\varphi(\cdot) \in L^1(\Omega)$ *and* $F_n(\omega) \to^h F(\omega)$ *μ-a.e. Then* $\Delta(E^{\Sigma_n} F_n, E^{\Delta} F) \to 0$ *as* $n \to \infty$ *and if* $\{E^{\Sigma_n} F_n\}_{n \geq 1}$ *is integrably bounded by* $\psi(\cdot) \in L^1(\ )$ *then* $E^{\Sigma_n} F_n(\omega) \to^h E^{\Sigma} F(\omega)$ *μ-a.e. as* $n \to \infty$.

Proof. The proof follows the exact same pattern as that of Theorem 4.1 and so is omitted. Q.E.D.

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